Jour. Ind. Soc. Ag. Statistics
Vol. XXXIX, No. 1 (1987), pp. 17-21

KRONECKER PRODUCT AND CONSTRUCTION OF BALANCED *N*-ARY DESIGNS*

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(Received : April, 1982)

SUMMARY

Kronecker product has been introduced formally for the construction of proper equireplicate balanced n-ary designs. The results are more general than those established by earlier authors. Earlier results are shown to be particular cases of the general result.

Keywords : Kronecker product; n-ary designs.

Introduction

Experimental situations often demand equal information on every elementary contrast. A design that meets this demand is said to be balanced. Assuming that the ith treatment occurs in the jth block n_{ij} times, $i = 1, 2, \ldots, v; j = 1, 2, \ldots, b$; $N = (n_{ij})$; $R = \text{diag}\ (r_1, r_2, \ldots, r_v)$; $K^{-1} = \text{diag}\ (K_1^{-1}, K_2^{-1}, \ldots, K_b^{-1})$ where $r_i = \sum_{j=1}^{b} n_{ij}$; $k_j = \sum_{i=1}^{v} n_{ij}$; and $C = R - NK^{-1}N^1$ a necessary and sufficient condition for the design N to be balanced is that C has all its diagonal elements equal and all off-diagonal elements also equal. If the design is proper (all the blocks have the same size k) and equireplicate it is easy to show that a necessary and sufficient condition for it to be variance balanced is that NN' is of the form $(h - \lambda)\ I(v) + \lambda E(v, v)$ where $E(v, v)$ is a $p \times q$ matrix with all elements unity and $I(v)$ is an identity matrix of order v.

*Part of the thesis submitted for the award of the degree of M.Sc. in Agricultural Statistics of the Kerala Agricultural University.

A design is said to be balanced n-ary if it is balanced and $(n - 1)$ is the maximum number of times a treatment occurs in a block and it is said to be incomplete if it has atleast one block which does not contain all treatments. The balanced n-ary designs were first introduced by Tocher [6]. The construction of these designs were later introduced by John [1], Murthy and Das [2], Nigam [3], Nigam *et al.* [4], Tyagi and Rizwi [7] and Surendran and Sunny [5]. The purpose of this paper is to introduce Kronecker Product of designs (Vartak, [8]) for the construction of n-ary designs from BIB designs and from existing p-ary designs.

2. Construction by Kronecker Product

The method of construction is contained in the two theorems that follow.

THEOREM 1. *If N_1 and N_2 are BIB designs with parameters $v, b_1, r_1, k_1, \lambda_1$ and $v, b_2, r_2, k_2, \lambda_2$ respectively and $N = a_1 E(1, b_2) \times N_1 + a_2 N_2 \times E(1, b_1)$ where a_1 and a_2 are positive integers and X stands for Kronecker Product N is a proper, equireplicate n-ary design, for $n = a_1 + a_2 + 1$.*

Proof. It is well known that

$$(A \times B)^1 = A^1 \times B^1 \text{ and } (A \times B)(A \times B)^1 = AA^1 \times BB^1.$$

Applying these results

$$\begin{aligned} NN' &= [a_1E(1, b_2)\, XN_1 + a_2N_2\, XE(1, b_1)]\, [a_1E(1, b_2)\, XN_1 \\ &\quad + a_2N_2\, XE(1, b_1)]^1 \\ &= a_1^2\, E(1, b_2)\, E(1, b_2)^1\, XN_1N_1^1 + a_2^2\, N_2N_2^1\, X\, E(1, b_1)\, E(1, b_1)^1 \\ &\quad + 2a_1a_2\, E(1, b_2)\, N_2^1\, XN_1\, E(1, b_1)^1 \end{aligned}$$

$$NN^1 = (r_i - \lambda_i)\, I(v) + \lambda_i\, E(v, v);\ \ i = 1, 2. \tag{2.1}$$

Hence

$$\begin{aligned} NN' &= [a_1^2\, b_2\, (r_1 - \lambda_1) + a_2^2\, b_1\, (r_2 - \lambda_2)]\, I(v) \\ &\quad + [a_1^2\, b_2\lambda_1 + a_2^2\, b_1\lambda_2 + 2a_1a_2r_1r_2]\, E(v.\ v.) \end{aligned} \tag{2.2}$$

Further the number of replications of each treatment is

$$r = a_1r_1b_2 + a_2r_2b_1 \tag{2.3}$$

and every block is of size

$$k = a_1k_1 + a_2k_2, \tag{2.4}$$

So that it is proper and equireplicate. Also a treatment can occur in a block at most $a_1 + a_2$ times and the design is incomplete. Therefore N is a proper equireplicate n-ary design if $a_1 + a_2 + 1 = n$.

The value of r can be alternatively established as follows :

The relation (2.2) implies that

$$0 = r - \frac{1}{k}\Big[a_1^2 b_2 (r_1 - \lambda_1) + a_2^2 b_1 (r_2 - \lambda_2) - v a_1^2 b_2 \lambda_1 - v a_2^2 b_1 \lambda_2 - 2 v a_1 a_2 r_1 r_2\Big]$$

On replacing $r_i - \lambda_i$ by $r_i k_i - v\lambda_i$ ($i = 1, 2$) and shifting r to the left hand side we get

$$\begin{aligned} r &= \frac{1}{k}\Big[a_1^2 b_2\, r_1 k_1 + a_2^2 b_1\, r_2 k_2 + 2 v a_1\, a_2 r_1 r_2\Big] \\ &= \frac{v r_1 r_2}{k}\left[\frac{a_1^2 k_1}{k_2} + a_2^2 \frac{k_2}{k_1} + 2 a_1 a_2\right] \\ &= \frac{v r_1 r_2}{k\, k_1 k_2} \quad (a_1 k_1 + a_2 k_2)^2 \\ &= \frac{v r_1 r_2}{k_1 k_2} (a_1 k_1 + a_2 k_2) \\ &= a_1 r_1 b_2 + a_2 r_2 b_1 \end{aligned}$$

which is same as (2.3).

Giving a_1 and a_2 particular values in theorem 1, some special cases are obtained.

We shall now consider the construction of proper equireplicate n-ary designs from known p-ary designs.

THEOREM 2. *If N_1 and N_2 are two proper balanced, equireplicate n_1-ary and n_2-ary designs in v treatments with b_1 and b_2 blocks respectively and a_1 and a_2 are two positive integers,*

$$N = a_1\, E(1, b_2)\, X\, N_1 + a_2 N_2\, X\, E(1, b_1)$$

is a n-ary balanced proper equireplicate design with $b_1 b_2$ blocks and $n = a_1(n_1 - 1) + a_2(n_2 - 1) + 1$.

Proof. Since N_1 and N_2 are balanced proper equireplicate designs $N_1 N_1^1$ and $N_2 N_2^1$ can be thrown into the form

$$N_1 N_1^1 = (h_1 - \lambda_1)\, I(v) + \lambda_1\, E(v, v)$$

$$N_2 N_2^1 = (h_2 - \lambda_2)\, I(v) + \lambda_2\, E(v \cdot v)$$

Now proceeding as in the case of theorem 1 it is easily seen that

$$NN^1 = a_1^2\, b_2\, [(h_1 - \lambda_1)\, I(v) + \lambda_1\, E(v,\, v)] + a_2^2\, b_1\, [(h_2 - \lambda_2)\, I(v)$$
$$+ \lambda_2\, E(v,\, v)] + 2a_1a_2r_1r_2\, E(v,\, v)$$

i.e. $NN^1 = (h - \lambda)\, I(v) + \lambda\, E(v,\, v)$ (2.6)

Where

$$h = a_1^2\, b_2h_1 + a_2^2\, b_1h_2 + 2a_1a_2r_1r_2 \quad (2.7)$$

$$= a_1^2\, b_2\lambda_1 + a_2^2\, b_1\lambda_2 + 2a_1a_2r_1r_2\, v$$

r_1 = number of replications of any treatment of N_1

r_2 = number of replications of any treatment of N_2

Further we note that if r is the number of replications of a treatment and k the size of block of N,

$$r = a_1b_2r_1 + a_2b_1r_2 \quad (2.8)$$
$$k = a_1k_1 + a_2k_2$$

k_i being the block size of N_i $(i = 1,\, 2)$.

3. Discussion

By taking $a_1 = 1$, $a_2 = 1$ in Theorem 1 we get a ternary design. This result was established by Nigam [3] by a different approach.

Putting $a_1 = 1$, $a_2 = n$ and $N_2 = I(v)$ a n-ary design with $b = b_1v$, $r = r_1v + nb_1$, $k = k_1 + n$ for v treatments is obtained. The result was first proved by an alternative approach by Tyagi and Rizwi [7].

If we take $a_1 = a_2 = 1$ and $N_1 = N_2$ a t-ary balanced equireplicate proper design, a $(2t - 1)$-ary design is obtained. This was first established by a different technique by Nigam [3]. The same author has also shown that given N_1 and N_2 as p-ary and s-ary proper equireplicate designs they can be used to construct $(p + s - 1)$-ary designs. Evidently this is a particular case of theorem 2.

ACKNOWLEDGEMENT

The authors wish to thank the referee for his valuable suggestions.

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